Schurman FLPERP Model - Giving a Perpetuity a Finite Life

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To perpetuity or not to perpetuity, that is the question! Is the terminal value of the security in question really a perpetuity or is there a significant probability that this perpetual cash stream will come to an abrupt end and therefore is technically not a perpetuity at all? A perpetuity is an annuity that has no end, or a stream of cash payments that continues forever. The terminal value of a security is the present value at some future point in time of all future cash flows when we expect a stable growth rate forever. In cases where there is a material probability that the perpetuity may not continue forever how do we reflect this probability in our terminal value calculation? In this paper we will (1) make a simple adjustment to the standard equation of a perpetuity to adjust for the probability of an abrupt termination at some unknown future point in time and (2) derive 'The Greeks' such that we can measure the sensitivity of the terminal value calculation to a change in assumptions.

A Hypothetical Case

Assume that we are engaged to value a company and we need to calculate a terminal value as of some future point in time T such that the terminal value calculation includes the present value of cashflows for the years $T+1$ and thereafter. We will assume that there is a material probability that the cash stream will end abruptly at some unknown time in the future. The assumptions for our terminal value calculation are...

C_T	=	\$100,000	=	Annualized free cash flow in the terminal value year T
g	=	5.00%	=	Annualized cash flow growth rate
k	=	20.00%	=	Annualized risk-adjusted discount rate
p	=	10.00%	=	Probability that the cash flow stream will end in any given year*

Question: What is the calculated terminal value (V_T) for our company?

* The probability of termination in each year is independent of the results from prior years. The probability of receiving cash in any year t is therefore $(1-p)^t$.

The Terminal Value Calculation

The standard terminal value equation ignores the possibility that the perpetuity is not perpetual. The terminal value of our company using the standard terminal value equation is...

$$V_T = \sum_{t=1}^{\infty} C_T \left[\frac{(1+g)}{(1+k)}\right]^t = \frac{C_T(1+g)}{k-g} = \frac{\$105,000}{0.20-0.05} = \$700,000 \tag{1}$$

We are given in the assumptions table above that the annual probability of the cash stream ending is 10%. In the year that the cash flow stream ends no cash is received at the end of that year or any subsequent year. How do we adjust the standard terminal value equation (1) above to reflect this new reality? The adjustment is actually pretty simple. If p is the probability that cash won't be received then $1-p$ is the probability cash will be received. The expected amount of cash that will be received in the first three years of the terminal value period is therefore...

Year	Cash Flow Equation	Expected Cash Received
1	$C_T(1+g)^1(1-p)^1$	$\$100,000 \times 1.05^1 \times 0.90^1 = \$94,500$
2	$C_T(1+g)^2(1-p)^2$	$\$100,000 \times 1.05^2 \times 0.90^2 = \$89,303$
3	$C_T(1+g)^3(1-p)^3$	$\$100,000 \times 1.05^3 \times 0.90^3 = \$84,391$

The revised equation for the terminal value of our company that reflects the probability of termination is (See

Appendix C)...

$$V_T = \sum_{t=1}^{\infty} C_T \left[\frac{(1+g)(1-p)}{(1+k)}\right]^t = C_T \left[\frac{(1+g)(1-p)}{(1+k)-(1+g)(1-p)}\right] = \$100,000 \times \left[\frac{(1.05)(0.90)}{(1.20)-(1.05)(0.90)}\right] = \$370,588 \tag{2}$$

Under the standard terminal value equation (1) the terminal value of our company is $700,000. Under the revised terminal value equation (2) the terminal value for our company is $370,588.

One question still remains - How do we get an estimate for the probability p? For this question there is no easy answer and will be based on circumstances particular to the valuation engagement. However, we can calculate a weighted average life of the terminal value period cash flows so as to get some comfort as to the value of p. The calculation for the weighted average life of the terminal value period cash flows is (See Appendix D)...

$$WAL = \frac{1}{p} = \frac{1}{0.10} = 10 \ \ years \tag{3}$$

The weighted average life calculated in equation (3) tells the appraiser that the expected amount of time that cash flows will be received during the terminal value period is 10 years. If the appraiser thinks that the weighted average life should be less then the probability p should be increased and visa versa.

Note that simply discounting the first 10 years of projected terminal value period cash flows and calling that number the terminal value is incorrect. This methodology assumes that there is a 0% probability of cessation in years 1 through 10 and a 100% probability of cessation in year 11 and accordingly front-loads cash flows such that the terminal value is overstated.

The Greeks

We want to measure the sensitivity of our dependent variable (terminal value) to changes to our independent variables (growth rate, discount rate and weighted average life). To do this we need the partial derivatives of our revised terminal value Equation (2) with respect to our independent variables. To facilitate the calculations we will make the following definitions...

$$G = 1+g \ \ ...and... \ \ K = 1+k \ \ ...and... \ \ P = 1-p = 1-\frac{1}{L} \, * \tag{4}$$

* Uses Equation (3) where L represents the weighted average life of the cash stream

Using the definitions above Equation (2) can be rewritten as...

$$V_T = C_T \frac{GP}{K-GP} \tag{5}$$

The partial derivative of Equation (5) with respect to G (one plus the growth rate) is...

$$\frac{\delta V_T}{\delta G} = C_T \left(\frac{\delta \, GP}{\delta G}(K-GP) - GP\frac{\delta \, (K-GP)}{\delta G}\right) \div (K-GP)^2 = \frac{C_T PK}{(K-GP)^2} \tag{6}$$

The partial derivative of Equation (5) with respect to K (one plus the discount rate) is...

$$\frac{\delta V_T}{\delta K} = C_T \left(\frac{\delta \, GP}{\delta K}(K-GP) - GP\frac{\delta \, (K-GP)}{\delta K}\right) \div (K-GP)^2 = \frac{-C_T GP}{(K-GP)^2} \tag{7}$$

The partial derivative of Equation (5) with respect to L (the weighted average life of the cash stream) is...

$$\frac{\delta V_T}{\delta L} = \frac{\delta V_T}{\delta P}\frac{\delta P}{\delta L} = C_T \left(\frac{\delta \, GP}{\delta P}(K-GP) - GP\frac{\delta \, (K-GP)}{\delta P}\right) \div (K-GP)^2 \times \frac{\delta(1-L^{-1})}{\delta L} = \frac{C_T GK}{L^2(K-GP)^2} \tag{8}$$

The table below presents the changes in terminal value given a change in assumptions:

Parameter	Current Value	Change	New Value	Change in Terminal Value
G	1.05	0.01	1.06	$\delta V_T = \frac{(100{,}000)(0.90)(1.20)}{(1.20-(1.05)(0.90))^2} \times 0.01 = 16,600$
K	1.20	0.01	1.21	$\delta V_T = \frac{(-100{,}000)(1.05)(0.90)}{(1.20-(1.05)(0.90))^2} \times 0.01 = -14,500$
L	10.0	1.00	11.0	$\delta V_T = \frac{(100{,}000)(1.05)(1.20)}{10^2(1.20-(1.05)(0.90))^2} \times 1.00 = 19,400$

Appendix

A) The sum of constant C that grows by theta in each period beginning with the current period and going to infinity is a geometric series with the following well known solution...

$$C\sum_{t=0}^{\infty}\theta^t = C\left[\frac{1}{1-\theta}\right] \text{ ...where... } \theta = \frac{(1+g)(1-p)}{(1+k)} \text{ ...and... } 0<\theta<1 \tag{9}$$

B) Adjust equation (9) above to start at period 1 rather than period zero.

$$\begin{aligned} C\sum_{t=1}^{\infty}\theta^t &= C\sum_{t=0}^{\infty}\theta^t - C\theta^0 \\ &= C\left[\frac{1}{1-\theta}\right] - C \\ &= C\left[\frac{1}{1-\theta} - 1\right] \\ &= C\left[\frac{\theta}{1-\theta}\right] \end{aligned} \tag{10}$$

C) If we define θ to be...

$$\theta = \frac{(1+g)(1-p)}{(1+k)} \tag{11}$$

Then the equation for terminal value (see equation in Appendix B) becomes...

$$\begin{aligned} V_T &= C_T\sum_{t=1}^{\infty}\theta^t \\ &= C_T\left[\frac{\theta}{1-\theta}\right] \\ &= C_T\left[\frac{\frac{(1+g)(1-p)}{(1+k)}}{1-\frac{(1+g)(1-p)}{(1+k)}}\right] \\ &= C_T\left[\frac{(1+g)(1-p)}{(1+k)-(1+g)(1-p)}\right] \end{aligned} \tag{12}$$

D) The equation for the weighted average life of cash flows (WAL) is...

$$\begin{aligned} WAL &= \sum_{t=1}^{n}\left[(1-p)^{t-1} - (1-p)^t\right] \times t \\ &= \sum_{t=1}^{n}(1-p)^t \times \left[(1-p)^{-1} - 1)\right] \times t \\ &= \frac{p}{1-p} \times \sum_{t=1}^{n} t\,(1-p)^t \end{aligned} \tag{13}$$

Note that the right side of the equation above is a low-order polylogarithm. The equation for weighted average life becomes...

$$\begin{aligned} WAL &= \frac{p}{1-p} \times (1-p)\left[\frac{1-(n+1)(1-p)^n + n(1-p)^{n+1}}{(1-(1-p))^2}\right] \\ &= \frac{p}{1-p} \times \frac{1-p}{p^2}\left[1-(n+1)(1-p)^n + n(1-p)^{n+1}\right] \\ &= \frac{1}{p} \times \left[1-(n+1)(1-p)^n + n(1-p)^{n+1}\right] \end{aligned} \tag{14}$$

As n goes to infinity the equation for weighted average life becomes...

$$\lim_{n \to \infty} WAL = \frac{1}{p} \tag{15}$$

Because...

$$\lim_{n \to \infty} (n+1)(1-p)^n = 0 \;\; ...and... \;\; \lim_{n \to \infty} n(1-p)^{n+1} = 0 \tag{16}$$